

Chapter 5: Rational Functions, Zeros of Polynomials, Inverse Functions, and Exponential Functions

QUIZ AND TEST INFORMATION: The material in this chapter is on **Quiz 5** and **Exam 4**. You should complete at least one attempt of Quiz 5 before taking Exam 4. This material is also on the final exam.

TEXT INFORMATION: The material in this chapter corresponds to the following sections of your text book: 3.3, 3.4, 3.6, 3.7, 4.2, and 4.3. Please read these sections and complete the assigned homework from the text that is given on the last page of the course syllabus.

LAB INFORMATION: Material from these sections is used in the following labs: Rational Functions and Exponential Functions. This information is also needed for future labs.

Rational Functions Exponential Functions

Assignments	3.3-3.4, 3.6-3.7, 4.2-4.3
Labs	Rational Functions Exponential Functions
Quiz	5
Test	4

Section 1: Basics of Rational Functions

- Definition: A rational function is a ratio of two _____, where the _____ is nonzero. In other words, a rational function is a function of the form $R(x) = \frac{\text{_____}}{\text{_____}}$, where _____ and _____ are polynomials with _____ nonzero.

Examples: Write some examples of rational functions.

- Example 1: Look at the examples you wrote above. What are the domains of these functions?

- Example 2: Let $R(x) = \frac{(x-2)(x+2)}{x-2}$. What is the domain of $R(x)$?

- Definition: A rational function is proper if _____

Otherwise the function is improper.

- Definition: An asymptote is _____

An asymptote is not part of the graph of a function, but it tells us information about the graph. We will talk about _____, _____, and _____ asymptotes.

- Other examples and notes:

Section 2: Vertical Asymptotes of Rational Functions

- Example 1: Consider the reciprocal function $R(x) = \frac{1}{x}$. Make a chart of the values of $R(x)$ for x -values 2, 1, $1/2$, $1/3$, $1/4$, and $1/5$. Next, make a chart of the values of $R(x)$ for x -values -2, -1, $-1/2$, $-1/3$, $-1/4$, and $-1/5$.

x	R(x)	x	R(x)
2		-2	
1		-1	
$1/2$		$-1/2$	
$1/3$		$-1/3$	
$1/4$		$-1/4$	
$1/5$		$-1/5$	

- Example 2: What is happening to the values of $R(x)$ as values of x get closer and closer to zero?

- Example 3: For $R(x) = \frac{1}{x}$, as the x -values approach 0, the y values become _____.
- Definition: A rational function $R(x)$ has a vertical asymptote at $x = r$ if as the x -values approach _____, the y -values become _____.

- Locating Vertical Asymptotes: A rational function $R(x) = \frac{p(x)}{q(x)}$ that is

in _____, will have a vertical asymptote at $x = r$ if _____.

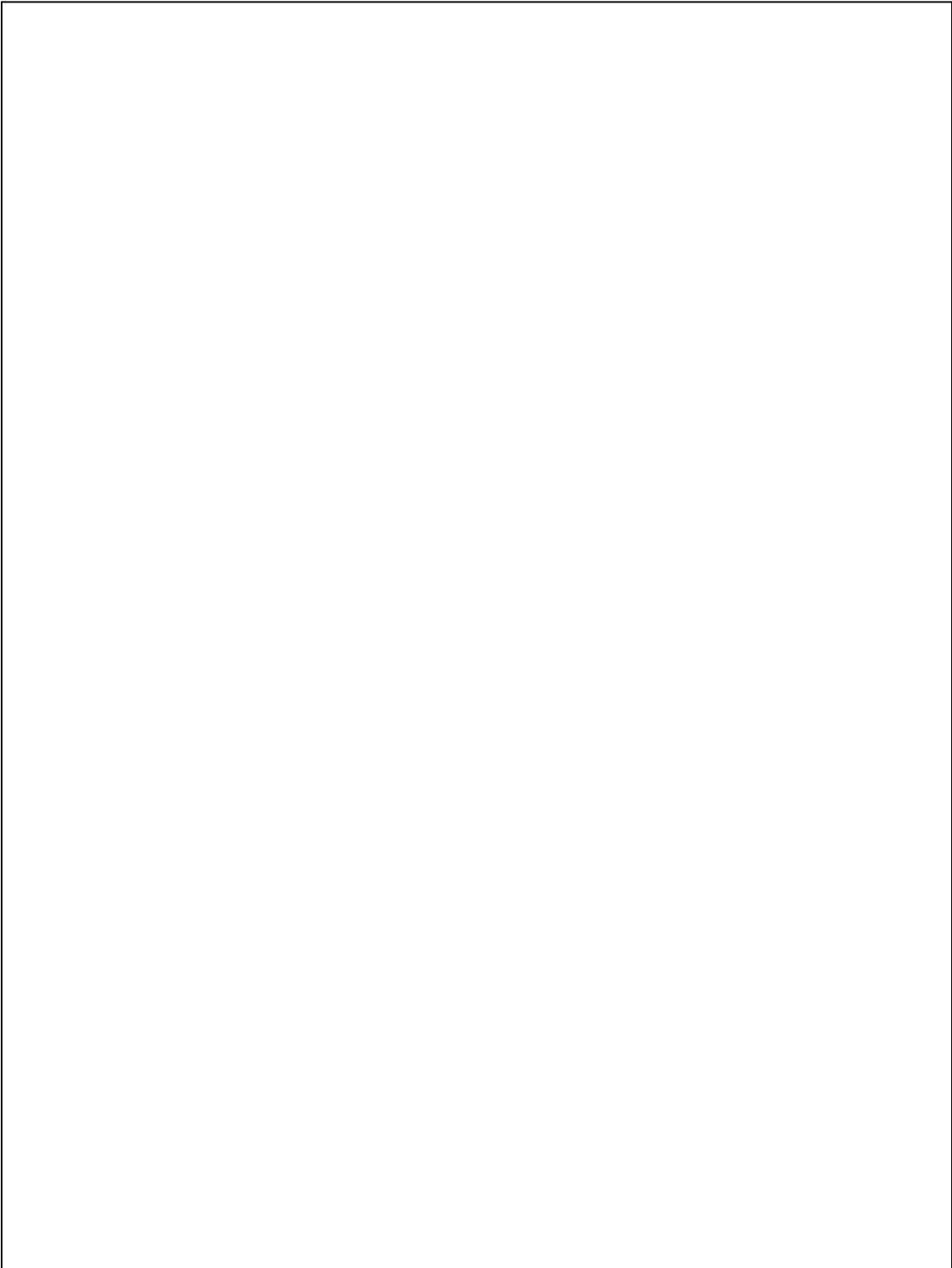
- Example 4: Where will $R(x) = (x - 1)^{-1}(x - 2)^{-1}$ have vertical asymptotes?

- Example 5: Will $R(x) = \frac{x^2 - 4x - 5}{x - 5}$ have any vertical asymptotes?

- Example 6: Can the graph of a rational function intersect its vertical asymptote?

- Example 7: How many vertical asymptotes can a rational function have?

- Other examples and notes:



Section 3: Horizontal and Oblique Asymptotes of Rational Functions

- Example 1: Consider the reciprocal function $R(x) = \frac{1}{x}$. Make a chart of the values of $R(x)$ for x-values 1, 2, 3, 4, 5, and 100. Next, make a chart of the values of $R(x)$ for x-values -1, -2, -3, -4, -5, and -100.

x	R(x)	x	R(x)
1		-1	
2		-2	
3		-3	
4		-4	
5		-5	
100		-100	

- Example 2: What is happening to the values of $R(x)$ as values of x get closer and closer to infinity or negative infinity?

- Example 3: For $R(x) = \frac{1}{x}$, as the x-values approach infinity or negative infinity, the y-values approach _____.
- Definition: A rational function, $R(x)$, has a horizontal asymptote at $y = b$ if, as the x-values approach positive or negative _____, the y-values approach _____. Horizontal asymptotes tell us about the e_____ b_____ of a function.

- Example 4: Can the graph of a rational function intersect its horizontal asymptote?

- Example 5: How many horizontal asymptotes can a rational function have?

- Example 6: Write some examples of rational functions that have horizontal asymptotes.

- Definition: If an asymptote is neither vertical nor horizontal, it is called o_____ or s_____. Oblique asymptotes also have to do with the e_____ b_____ of a function.

- Example 7: What is an example of a function with an oblique asymptote?

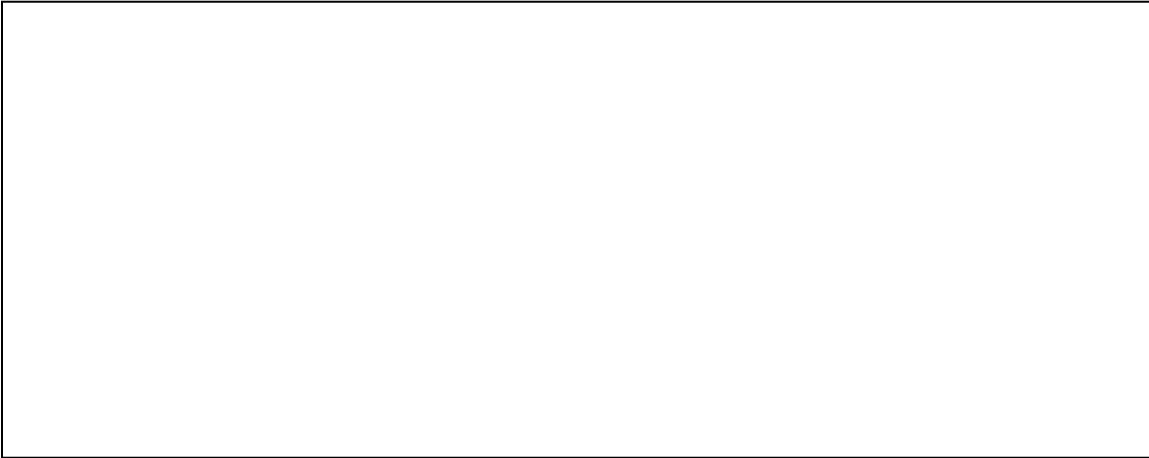
- Example 8: Can a graph intersect its oblique asymptote?

- Example 9: How many oblique asymptotes can a rational function have? Can a rational function have both oblique and horizontal asymptotes? Can a rational function have oblique and vertical asymptotes?

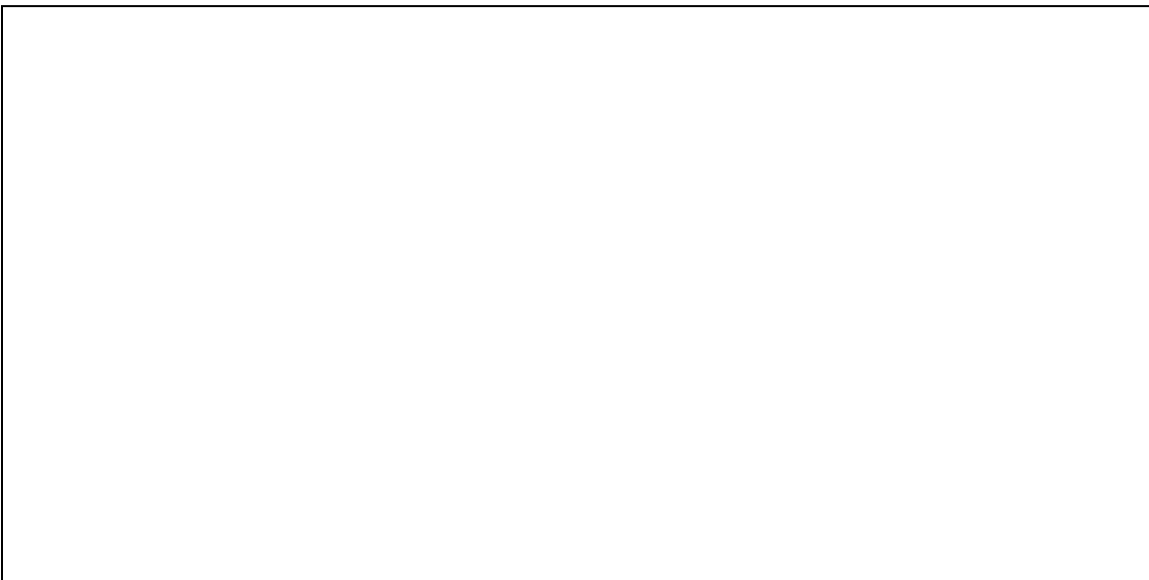
- Example 10: Explore what happens when you divide one polynomial, $p(x)$, by another polynomial, $q(x)$, when:
 1. The degree of $p(x)$ is less than the degree of $q(x)$:

2. The degree of $p(x)$ is equal to the degree of $q(x)$:

3. The degree of $p(x)$ is one more than the degree of $q(x)$:



4. The degree of $p(x)$ is more than one more than the degree of $q(x)$:



- Locating Horizontal and Oblique Asymptotes: Let

$$p(x) = a_n x^n + \dots + a_1 x + a_0 \text{ and } q(x) = b_m x^m + \dots + b_1 x + b_0$$

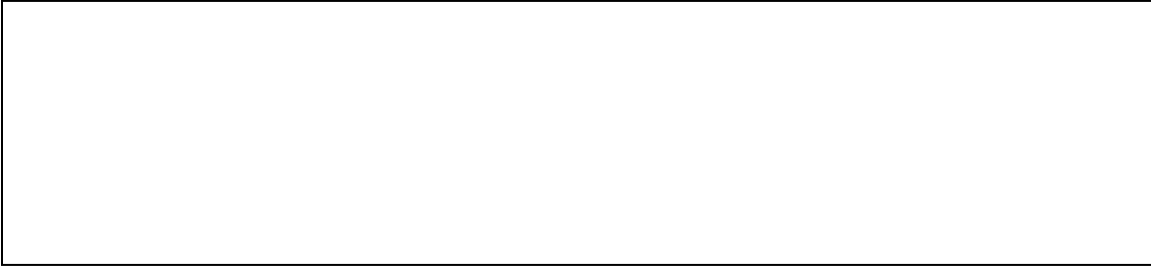
be two nonzero polynomials. Fill in this chart about finding the

horizontal or oblique asymptotes of the rational function $R(x) = \frac{p(x)}{q(x)}$.

**HOW TO LOCATE HORIZONTAL OR OBLIQUE ASYMPTOTES OF A
RATIONAL FUNCTION, $R(x) = p(x)/q(x)$**

Degree of the numerator, $p(x)$, versus degree of the denominator, $q(x)$	Type of Asymptote (horizontal or oblique)	How to Find Asymptote	Example
$R(x)$ is proper (the degree of $p(x)$ is less than the degree of $q(x)$) ($n < m$)			
The degree of $p(x)$ is equal to the degree of $q(x)$ ($n = m$)			
The degree of $p(x)$ is one more than the degree of $q(x)$ ($n = m + 1$)			
The degree of $p(x)$ is more than one more than the degree of $q(x)$ ($n > m + 1$)			

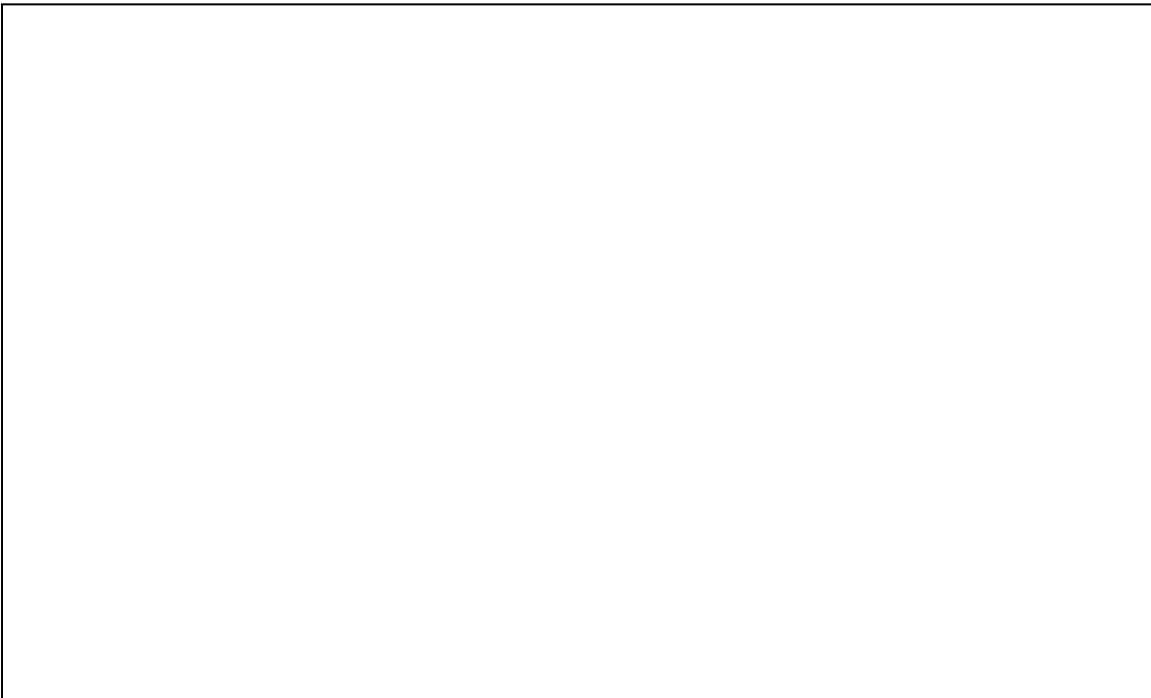
- Example 11: Find the horizontal or oblique asymptotes, if any, of the graph of $(4x - 1)/(x^2 + 2x - 1)$.



- Example 12: Find the horizontal or oblique asymptotes, if any, of the graph of $(5x^2 + 4x - 1)/(17x^2 + 2x - 1)$.



- Example 13: Find the horizontal or oblique asymptotes, if any, of the graph of $(2x^3 - 5x^2 + 4x - 1)/(x^2 + 2x - 1)$.



- Example 14: Find the horizontal or oblique asymptotes, if any, of the graph of $(2x^4 - 5x^2 + 4x - 1)/(x^2 + 2x - 1)$.

- Example 15: Which of the following statements is true about $R(x)$ if

$$R(x) = \frac{(x-2)(x+1)}{(x-5)(x+2)}$$

1. R has 2 x-intercepts, 2 vertical asymptotes, and 2 horizontal asymptotes
2. R has no x-intercepts, 2 vertical asymptotes, and 2 horizontal asymptotes
3. R has 2 x-intercepts, 2 vertical asymptotes, and 1 oblique asymptote
4. R has 2 x-intercepts, 2 vertical asymptotes, and 1 horizontal asymptote
5. None of these.

- Other examples and notes:

Section 4: Complete Graphs of Rational Functions

- Note: To make a complete graph of a rational function, $R(x)$, we will consider the following:
 1. the domain of $R(x)$,
 2. the simplified form of $R(x)$,
 3. the intercepts of $R(x)$ (if any),
 4. the asymptotes of $R(x)$ (if any), and
 5. a sign chart or test points to find the shape of the graph.

- Note: To find the y-intercepts, if any, of a rational function: _____
_____.

To find the x-intercepts, if any, of a rational function: _____
_____.

- Example 1: Find the x and y-intercepts, if any, of $f(x) = (x^2 + 2x)/(x^2 + 7x - 6)$.

x-intercepts:

y-intercepts:

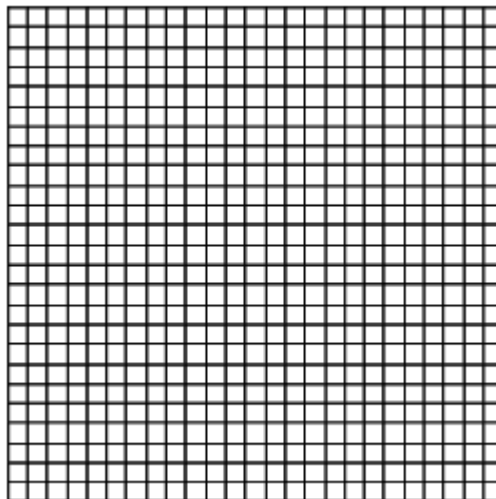
- Example 2: Suppose $p(x)$ is a polynomial. Where can the sign of $p(x)$ change? In other words, where can $p(x)$ change from being positive (above the x -axis) to negative (below the x -axis)?

- Making a sign chart: Make a number line and label the x -intercepts of the numerator and denominator on the number line. Test a point in each region to see if the function is positive or negative there (this is similar to what we did with polynomials).



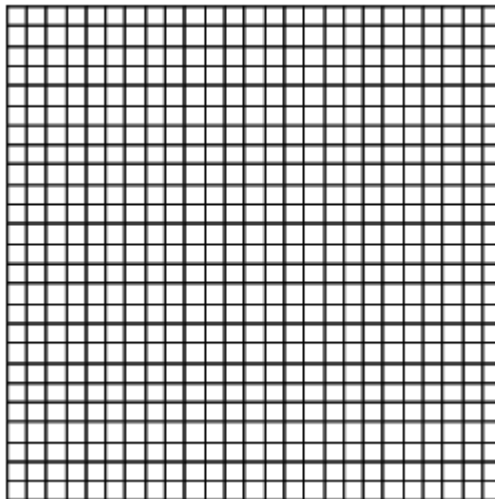
- Example 3: Make a complete graph of $f(x) = \frac{x}{(x-1)(x+2)}$.

1. domain:
2. simplified form:
3. intercepts:
4. asymptotes:
5. sign chart:



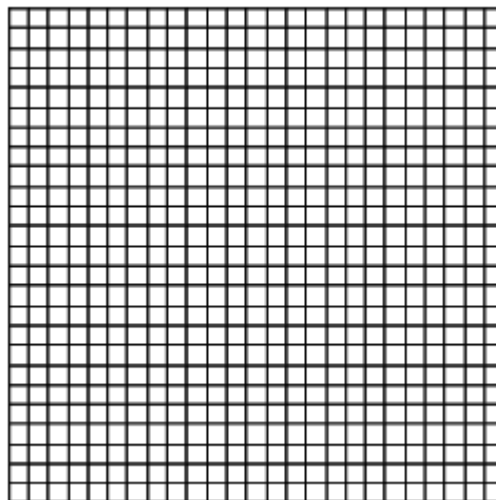
- Example 4: Make a complete graph of $G(x) = \frac{2x+4}{x-1}$.

1. domain:
2. simplified form:
3. intercepts:
4. asymptotes:
5. sign chart:



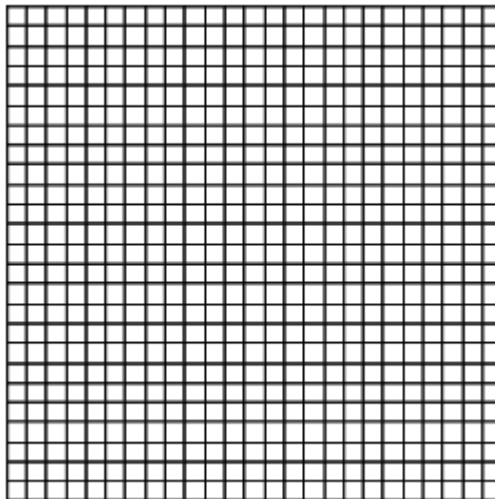
- Example 5: Make a complete graph of $R(x) = \frac{3x^2(x-1)}{(x+4)(x-3)}$.

1. domain:
2. simplified form:
3. intercepts:
4. asymptotes:
5. sign chart:



- Example 6: Make a complete graph of $f(x) = \frac{x(x-1)}{x}$.

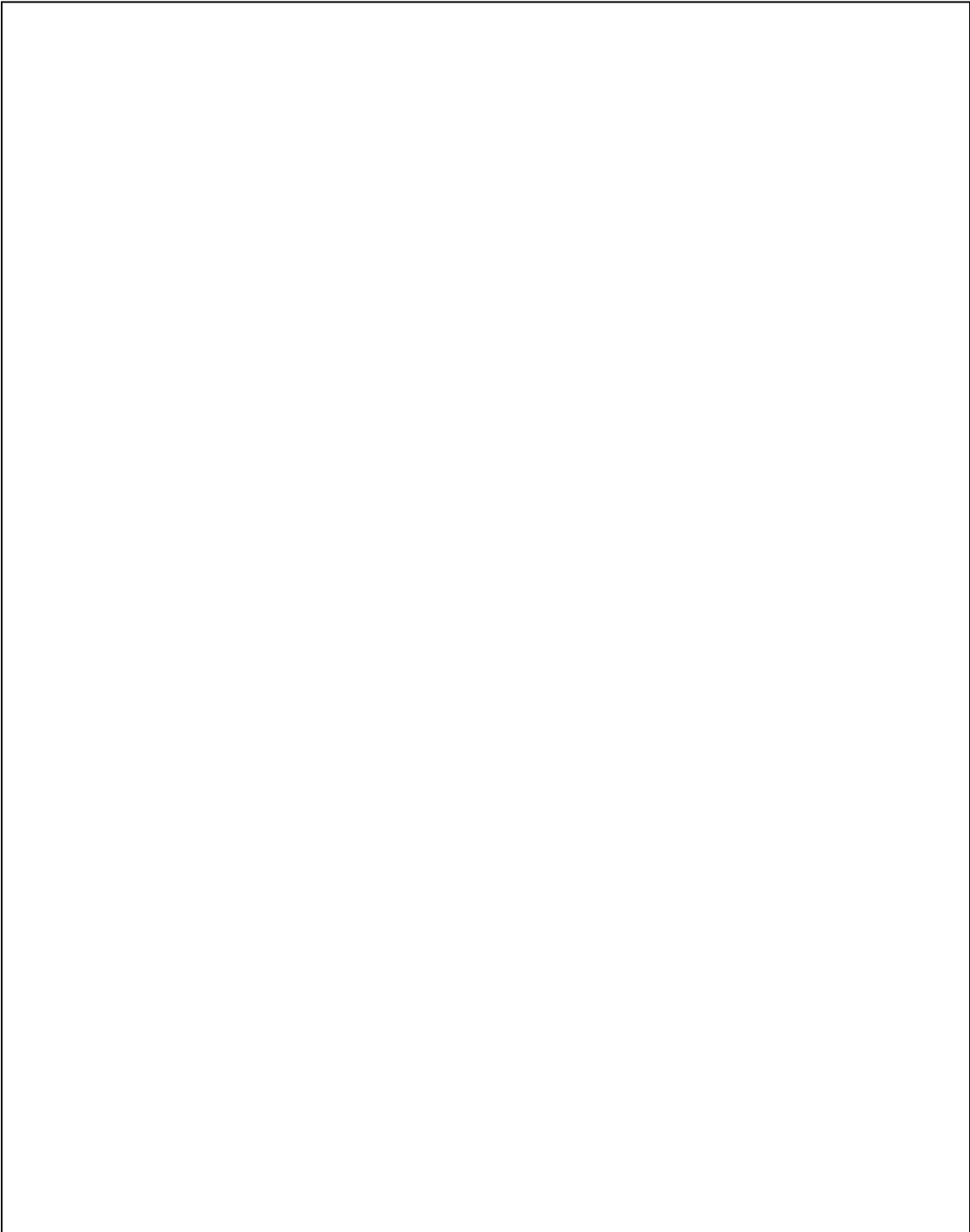
1. domain:
2. simplified form:
3. intercepts:
4. asymptotes:
5. sign chart:



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Chapter 5: Rational Functions, Zeros of Polynomials, Inverse Functions, and Exponential Functions

- Other examples and notes:



Section 5: Real Zeros of Polynomials and the Fundamental

Theorem of Algebra

- The Remainder Theorem: If a polynomial $f(x)$ is divided by $(x-k)$, then the remainder is $r = f(\text{_____})$.
- Note: For a polynomial, $f(x)$ (here $f(x)$ is called the dividend):
 $f(x) = (d\text{_____}) (q\text{_____}) + r\text{_____}$.
Suppose a polynomial $f(x)$ is divided by $(x - k)$. Thus,
 $f(x) = (x - k) q(x) + r$, for some remainder _____ and some quotient _____. Then $f(k) = (k - k)q(k) + r = r$.
- Example 1: Find the remainder if $f(x) = 5x^3 + 6x + 8$ is divided by $x + 2$.

- The Factor Theorem: A polynomial $f(x)$ has a factor of $(x - k)$ if and only if $f(k) = \text{_____}$. In other words, the remainder when you divide by $(x - k)$ is _____.

- Example 2: Is $(-4, 0)$ an x-intercept of $f(x) = x^3 - 28x - 48$? Place the polynomial in factored form.

- Example 3: Are the following factors of $f(x) = x^3 - 4x^2 + 5x - 2$?
 1. $x-2$ _____
 2. $x+2$ _____
 3. $x+3$ _____

- The Rational Zero Theorem: Let f be a polynomial of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where a_n and a_0 are not zero and each coefficient is an integer. If p/q , in lowest terms is a rational zero of f , then p must be a factor of a_0 and q a factor of a_n .

- Example 4: Let $f(x) = x^3 + 4x^2 + x + 4$. What are possible values of p ? What are possible values of q ? What are the possible rational roots?

Possible values of p :

Possible values of q :

Possible rational roots (all possible combinations of p/q):

- Example 5: Find the rational zeros of the polynomial. List any irrational zeros correct to two decimal places. $f(x) = x^3 + 4x^2 + x + 4$

- Example 6: List the potential rational zeros of the following polynomial function. $F(x) = 3x^3 + 3x^2 - 5x + 1$

Possible values of p:

Possible values of q:

Possible rational roots (all possible combinations of p/q):

- Example 7: List the potential rational zeros of the polynomial function. Do not find the zeros. $F(x) = 5x^3 + 17x^2 + 2x + 2$.

Possible values of p:

Possible values of q:

Possible rational roots (all possible combinations of p/q):

- Example 8: Factor completely. $f(x) = x^3 - 3x + 2$

- Example 9: Can we factor $f(x) = x^2 + 4$ over the real numbers?

- Definition: A polynomial is called irreducible or prime if it cannot be _____ over the real numbers.
- The Fundamental Theorem of Algebra: Every complex polynomial of degree n greater than or equal to one has at least one complex _____.

- The Conjugate Pairs Theorem: Let $f(x)$ be a complex polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f , then _____ is also a zero.
- Example 10: Find the remaining zeros of $f(x) = x^3 + 4x^2 + x + 4$ given that one zero is at i .

- Example 11: Find the remaining zeros of a polynomial with real number coefficients whose degree is 3 and has zeros at 5 and $2-3i$.

- Intermediate Value Theorem: Suppose f is a _____ function and $a < b$. If $f(a)$ and $f(b)$ are of opposite sign, then f has _____ between a and b .

- Example 12: Use the Intermediate Value Theorem to show that the following polynomial has at least one zero in the given interval.

$$f(x) = x^5 + 2x^4 + 3x^2 - 5 \text{ in } [0,1]$$

- Other examples and notes:

Section 6: One-to-One and Inverse Functions

- Example 1: In the following two examples, find the inverse of the given functions.

1. Suppose Bob, Mary, and Susan enter a contest where they will each win one different prize.

$\{(\text{Bob}, \$100), (\text{Mary}, \$200), (\text{Susan}, \$300)\}$

$\{(\quad , \quad), (\quad , \quad), (\quad , \quad)\}$

2. Suppose Bob, Mary, and Susan enter a contest where more than one person could win the same prize.

$\{(\text{Bob}, \$100), (\text{Mary}, \$100), (\text{Susan}, \$300)\}$

$\{(\quad , \quad), (\quad , \quad), (\quad , \quad)\}$

- Example 2: In the proceeding example, are the inverses functions?

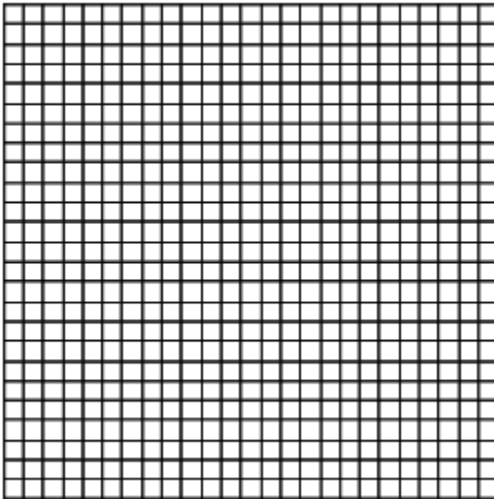
- Definition: A function is _____ if, whenever a and b are two distinct elements in the domain of f, f(a) is not equal to f(b). Another way to think about this is that a function f is one-to-one if its inverse is also a _____.

- Definition: Graphically, to check if a function is one-to-one use the H_____ L_____ T_____:

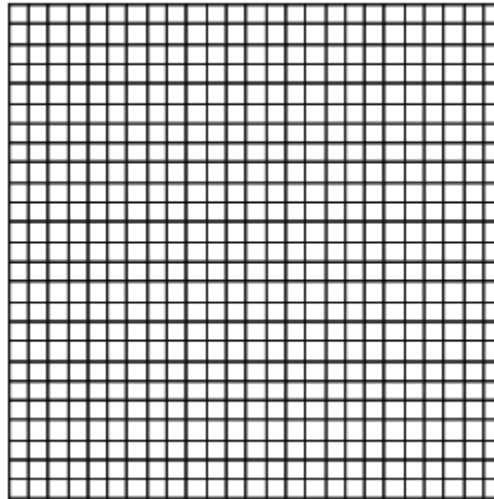
If every horizontal line intersects the graph of a function f in at most one point, then f is _____.

- **Example 3:** Sketch the graphs of the following functions and use the horizontal line test to decide if the functions are one-to-one.

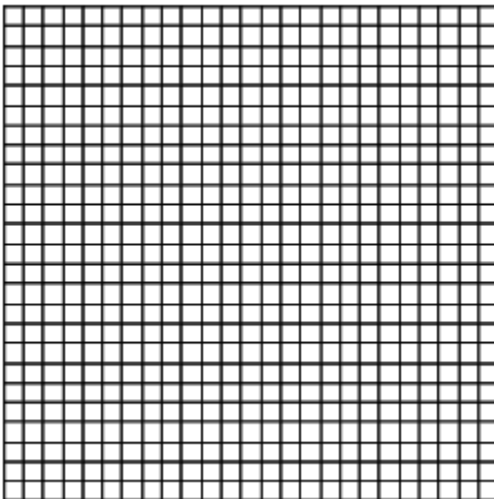
1. $y = x^3$ _____



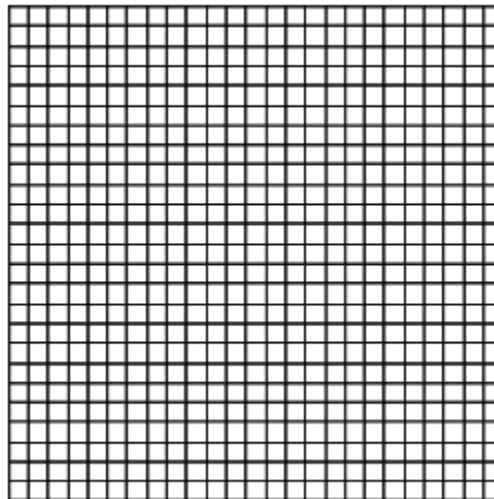
2. $y = x^2$ _____



3. $y = 2x$ _____

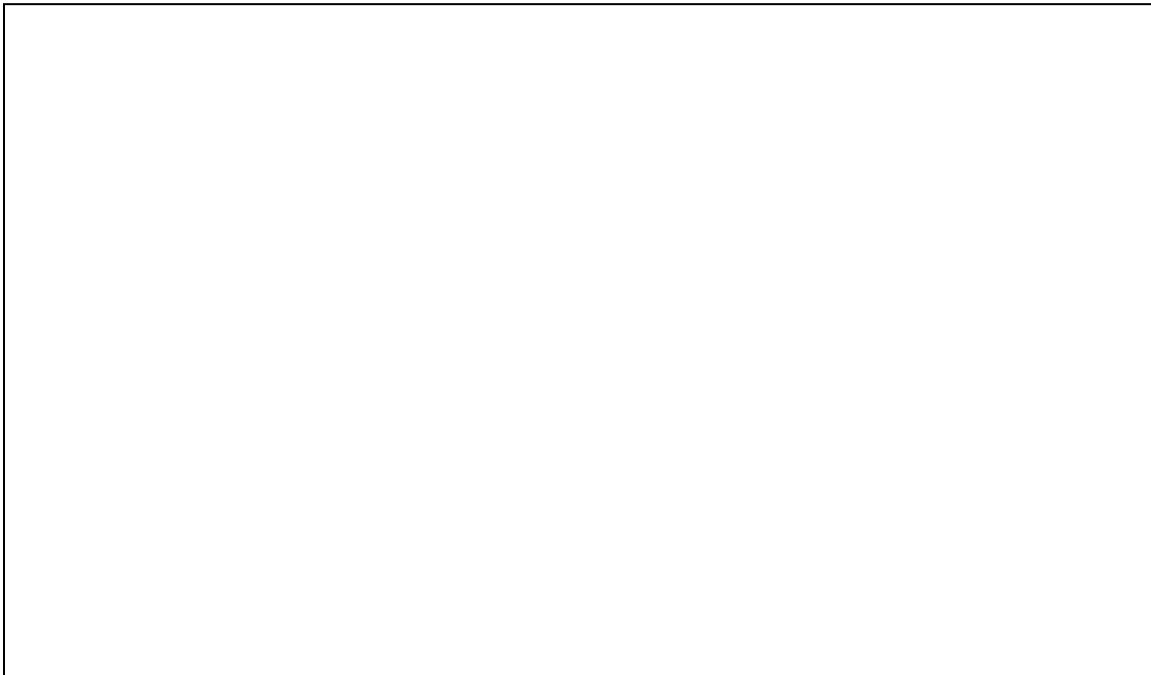


4. $y = 1/x$ _____



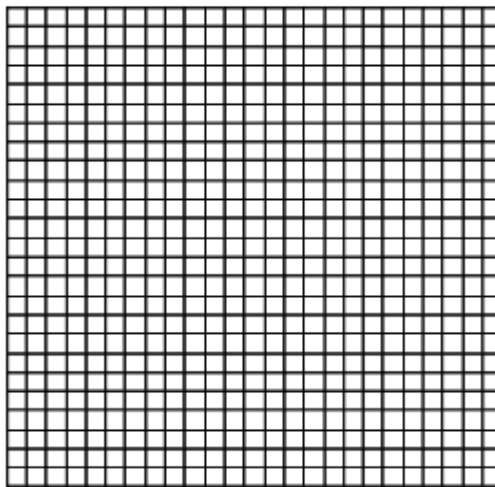
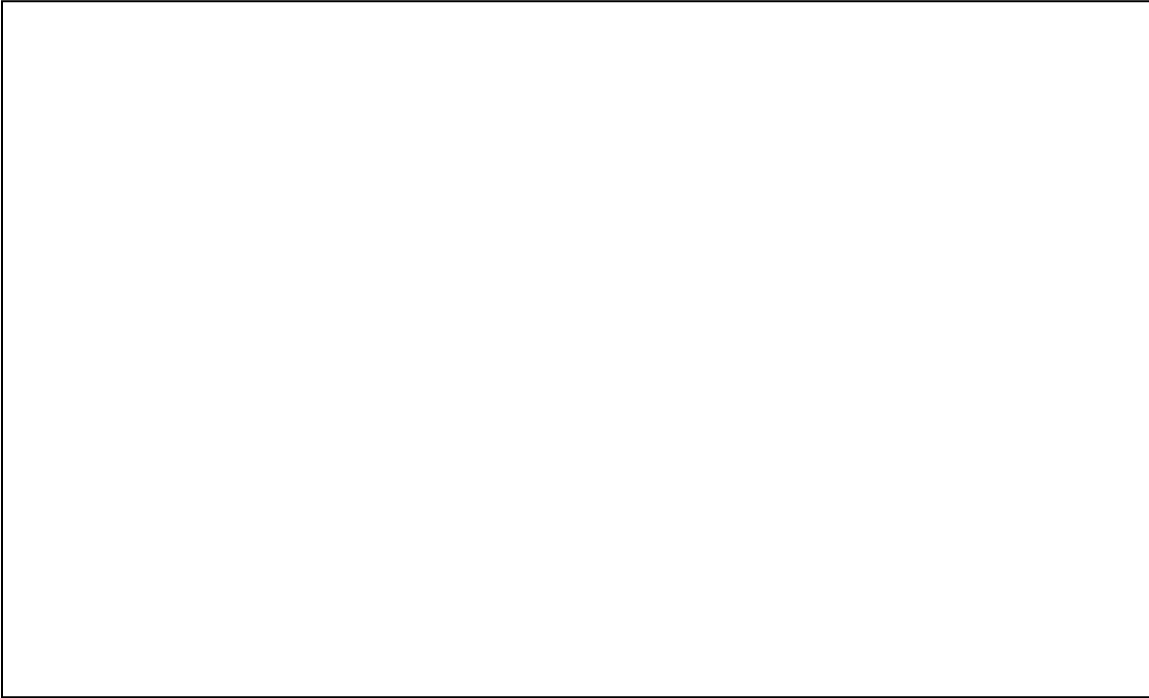
- Notation: Given a one-to-one function $f(x)$, we denote its inverse _____ . The domain of $f(x)$ is the _____ of $f^{-1}(x)$, and the range of $f(x)$ is the _____ of $f^{-1}(x)$.
- Note: For a function $f(x)$ and its inverse function $f^{-1}(x)$:
 $f(f^{-1}(x)) = \underline{\hspace{2cm}}$ and $f^{-1}(f(x)) = \underline{\hspace{2cm}}$.
- Example 4: Decide whether or not the functions are inverses of each other. $F(x) = 2x + 6$; $G(x) = 1/2 x - 3$

- Note: Algebraically, to find the inverse function of a one-to-one function $f(x)$ follow the following procedure:
 1. Write the function as $y = f(x)$.
 2. Switch _____ and _____. In other words, wherever there is an _____, write _____, and where there is a _____ write _____. The function should now look like $x = f(y)$.
 3. Solve for _____.
 4. Check that the composition of the two functions is _____.
- Example 5: The function $f(x) = (2x + 1)/(x - 1)$, x not equal to one, is one-to-one. Find its inverse and check the result.

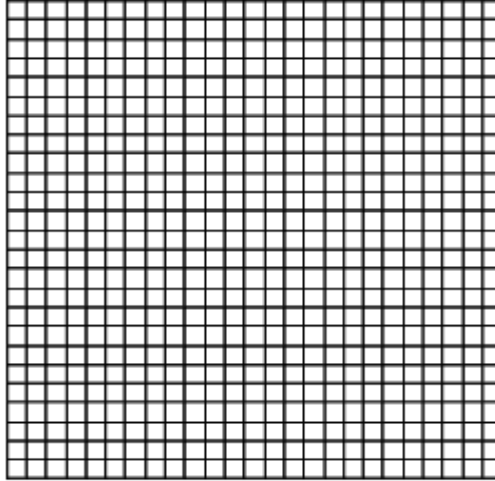


- Note: Given the graph of a one-to-one function $f(x)$, to graph the inverse function, note that the graph of f and f^{-1} are symmetric to each other with respect to the line $y = \underline{\hspace{2cm}}$. If (a,b) is on the graph of a one-to-one function f , then $(\underline{\hspace{1cm}}, \underline{\hspace{1cm}})$ is on the graph of its inverse.

- Example 6: Sketch the graph of $f(x) = (x - 5)/(2x + 3)$. Using this graph, sketch the graph of $f^{-1}(x)$.



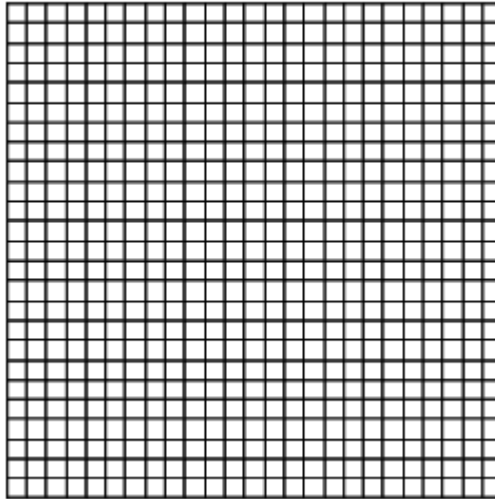
- Example 7: Sketch the graph of $f(x) = x^2$. Using this graph, sketch the graph of $f^{-1}(x) = x^{1/2}$.



- Other example and notes:

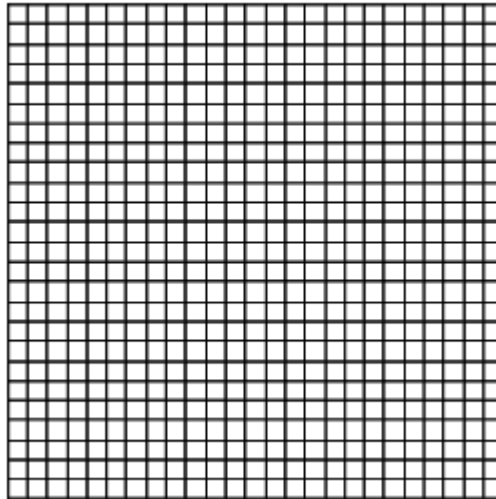
Section 7: Exponential Functions

- Definition: An exponential function is a function of the form _____, where _____ is a positive real number not equal to one. We call _____ the base of the function.
- Note: The domain of an exponential function is _____.
- Example 1: Sketch the graph of $f(x) = 2^x$, which is an exponential function.



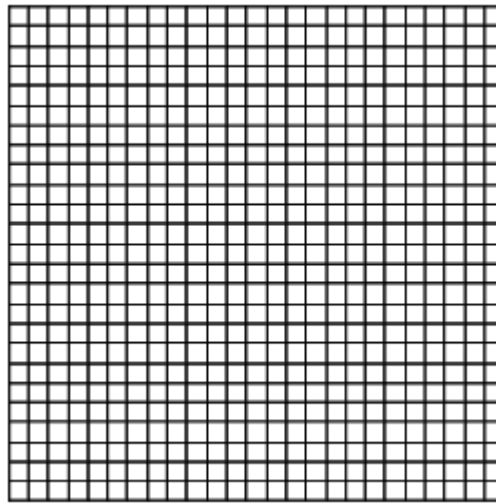
- Properties of Graphs of Exponential Functions: For $f(x) = a^x$ where $a > 1$, the following are properties of the graph of $f(x)$:
 1. The function is i_____.
 2. The graph includes the points (____, 1) and (1, a).
 3. As x approaches infinity, $f(x)$ approaches _____.
 4. As x approaches negative infinity, $f(x)$ approaches _____.
 5. Domain = _____
 6. Range = _____

- Example 2: Sketch the graph of $f(x) = (1/2)^x$, which is an example of an exponential function.

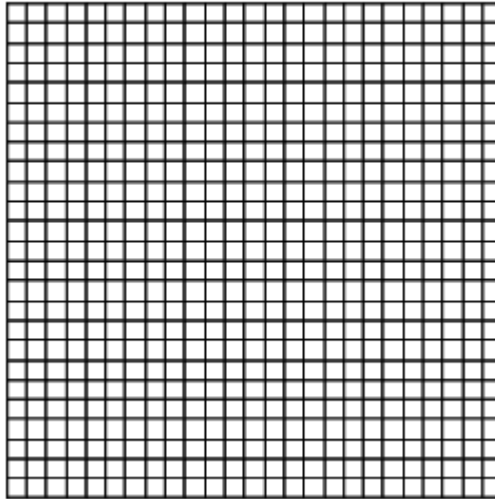


- Properties of Graphs of Exponential Functions: For $f(x) = a^x$ where $0 < a < 1$, the following are properties of the graph of $f(x)$:
 1. The function is d_____.
 2. The graph includes the points (____, 1) and (1, a).
 3. As x approaches infinity, $f(x)$ approaches _____.
 4. As x approaches negative infinity, $f(x)$ approaches _____.
 5. Domain = _____
 6. Range = _____

- Example 3: Rewrite 2^{-x} without a negative exponent. How do the graphs of 2^x and 2^{-x} compare?



- Example 4: Sketch the graph of $y = e^x$, $y = e^{-x}$, and $y = e^x + 1$ on the same set of axes and label the graphs. Give the domain, range, and horizontal asymptote for each.



Function	Domain	Range	Horizontal asymptote
$y = e^x$			
$y = e^{-x}$			
$y = e^x + 1$			

- Other examples and notes: